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# Amplification of Gravitational Waves in Scalar-Tensor Inflationary Lambda-Model

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**Abstract** After reviewing the scalar-tensor inflationary solutions by Berman and Trevisan (Int. J. Theor. Phys. 29, 1411–1414, 2009), we obtain solutions for the amplification of gravitational waves in the models. The solutions consider a perfect gas equation of state, with cosmic pressure proportional to the energy density, the proportionality constant being smaller than $-2/3$, and a cosmological term.

**Keywords** Gravitational waves · Scalar-tensor · Inflation

## 1 Introduction

The purpose of this paper is to show that, in the scalar-tensor gravity theory for the inflationary phase, in a lambda- Universe, there is amplification of gravitational waves. Berman [8], considered gravitational waves amplified in a static Universe. The theory of gravitational waves, has been dealt in several books, so we will not introduce this subject. As to scalar-tensor gravity, we refer to the books by Berman [5], Fujii and Maeda [16] and Faraoni [15]. Inflation is considered by Linde [17], and a modern review is given by Weinberg [20].

In the next section, we present an inflationary solution in scalar-tensor gravity, after Berman and Trevisan [7, 10]. Then (Sect. 3), we consider the amplification of gravitational waves in this model. Section 4 gives concluding remarks.

As Corda [14] has posed in his famous essay, interferometric detection of gravitational waves may be the definitive test for General Relativity or other scalar-tensor theories. Also, the same author has considered the primordial production of massive relic gravitational waves in a weak modification of General Relativity [13]. The production of stochastic background of relic gravitational waves, using the so-called adiabatically-amplified zero-point fluctuation process has produced a proof that, in principle, inflationary scenario provides distinctive spectra of gravitational waves.

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The scalar-tensor gravity analysis of stochastic background of relic scalar gravitational waves, has been treated recently by Capozziello et al. [12].

## 2 Review of Scalar-Tensor Inflationary Model [10]

New evidence for primordial inflation has been recently gathered through cosmic microwave observation [20]. Barrow [1] has pointed out the possible relevance of scalar-tensor gravity theories in the study of the inflationary phase during the early Universe. He obtained exact solutions for homogeneous and isotropic cosmologies in vacuum and radiation cases, for a variable coupling "constant", $\omega = \omega(\phi)$, where $\phi$ stands for the scalar field. For accounts on inflation, see, for instance, Linde's book [17].

Berman and Trevisan [7, 10] have extended Barrow's paper by the study of an inflationary exponential phase. Their calculation can be considered also as a complement to Berman and Som's paper [9] dealing with the inflationary phase in B.D. original framework, which was followed by a letter by Berman [4] where he studied the same problem in the context of a B.D. theory endowed with a cosmological constant. For scalar-tensor theories, consult the books by Berman [5], Faraoni [15], and Fujii and Maeda [16]. In Berman [6], we find a *rationale* for the existence of a cosmological "constant", though we must remember that a negative cosmic pressure may be also responsible for accelerated expansion, which includes exponential inflation.

### 2.1 The Field Equations

One way to formulate a scalar-tensor theory of gravitation can be cast with the following Lagrangian:

$$L_\phi = -\phi R + \phi^{-1}\omega(\phi)\partial_a\phi\partial^a\phi + 16\pi L_m - 2\Lambda(\phi), \tag{1}$$

where $L_m$ is the Lagrangian for matter fields, and $\phi$ is the scalar field. If $\omega = \text{const}$ we obtain the Brans-Dicke [11] theory. This Lagrangian was adopted by Barrow and Maeda [2]. For a discussion about the Lagrangians of the scalar theories of gravitation, see Liddle and Wands [18]. The cosmological term $\Lambda(\phi)$ is taken also to mean time-dependent lambda.

By varying the action associated with (1) with respect to the space-time metric and the scalar field $\phi$, respectively we obtain the generalized Einstein equations and the wave equation for $\phi$ [1]:

$$G_{ab} = -\frac{8\pi}{\phi}T_{ab} - \frac{\omega}{\phi^2}\left[\phi_a\phi_b - \frac{1}{2}g_{ab}\phi_i\phi^i\right] - \frac{1}{\phi}[\phi_{a;b} - g_{ab}\Box\phi] - \frac{\Lambda}{\phi}g_{ab}, \tag{2}$$

$$[3+2\omega]\Box\phi = 8\pi T - \left(\frac{d\omega}{d\phi}\right)\phi_i\phi^i + 2\phi\frac{d\Lambda}{d\phi} - 4\Lambda. \tag{3}$$

In General Relativity theory, in face of a perfect fluid matter field, from the field equations, it is derived the energy momentum conservation law,

$$T^{ab}_{;b} = 0. \tag{4}$$

In Brans-Dicke theory, Weinberg [19] has commented that in order to preserve the Principle of Equivalence, the scalar-field does not enter into the conservation equation above, which takes into consideration only the matter-fields. For scalar-tensor theories, as well, this conservation equation is imposed on the same token, but, of course, if we take the field equations, say, for Robertson-Walker's metric, obtaining an equation for cosmic pressure and other for the energy density, we could combine those equations, along with the scalar-field one, and obtain a generalisation of the kind,

$$G^{ab}_{;b} = 0,$$

where the conservation law applies to the right-hand-side of (2).

With Robertson-Walker's metric,

$$ds^2 = dt^2 - a^2\left[\left(1 - kr^2\right)^{-1} dr^2 + r^2 d\theta + r^2 \sin^2\theta d\varphi^2\right] \tag{5}$$

we find, from (4), (2) and (3), that:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi\rho}{3\phi} - \frac{\dot{\phi}}{\phi}\frac{\dot{a}}{a} + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2} + \frac{\Lambda}{3\phi}, \tag{6}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \tag{7}$$

$$\ddot{\phi} + \left[3\frac{\dot{a}}{a} + \frac{\dot{\omega}}{2\omega + 3}\right]\dot{\phi} = \frac{1}{3 + 2\omega}\left[8\pi(\rho - 3p) - 2\phi\frac{\dot{\Lambda}}{\dot{\phi}} + 4\Lambda\right], \tag{8}$$

where overdots stand for time derivatives.

Consider the solution for $\phi(t)$, and $\omega(t)$:

$$\frac{\dot{a}^2}{a^2} = -\frac{\dot{\phi}}{\phi}\frac{\dot{a}}{a}, \tag{9}$$

and,

$$\frac{8\pi\rho}{\phi} = -\frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2} - \frac{\Lambda}{3\phi}. \tag{10}$$

By summing (9) and (10), we recover expression (6), so that the above two equations, result in a particular solution, though that it has some generality altogether.

With the above field equations, that they yield an interesting inflationary solution, whereby the scale-factor is exponential. The solution is the following,

$$a = a_0 e^{Ht}, \tag{11}$$

where $a_0$, $H$, are constants, and

$$p = \alpha\rho \quad (\alpha = \text{const}). \tag{12}$$

In General Relativity, $\alpha = -1$ for the inflationary phase; here, we must consider also other possibilities. Law (12) stands for a "perfect" gas equation of state. From (7) and (11), we find, using (12),

$$\rho = \rho_0 e^{-3H(1+\alpha)t} \tag{13}$$

where $\rho_0 = \text{const}$.

Remember that $H = \dot{a}/a$ stands for Hubble's parameter.

We find from (9),

$$\phi(t) = \phi_0 e^{-Ht} \tag{14}$$

with $\phi_0 = \text{const}$.

From (10), we get a possible solution with, $\omega \gg 3/2$,

$$\omega \cong \omega_0 e^{-H(2+3\alpha)t} \tag{15}$$

with $\omega_0 =$ positive constant, and $\phi_0 > 0$.

From (9) and (10), we find,

$$\Lambda = \Lambda_0 e^{-3H(1+\alpha)t} \quad (\Lambda_0 = \text{const}),$$

and,

$$8\pi\rho_0 + \frac{1}{3}\Lambda_0 + \frac{1}{6}H^2\phi_0\omega_0 = 0.$$

It is then, highly desirable that $\omega$ grow with time, so we impose,

$$\alpha < -\frac{2}{3}. \tag{16}$$

This condition on the equation of state encompasses the case $\alpha = -1$ of G.R.

## 3 Amplification of G.W.'s in the Above Model

Barrow et al. [3], laid the background for studying the amplification of gravitational waves in scalar-tensor gravity. Considering a variable cosmological term, coupling "constant", etc., the weak gravitational waves are represented by, the perturbations $h_{\mu\nu}$:

$$h_{ij}(t, \mathbf{x}) = \int d^3k h_{ij}^{(k)}(t, \mathbf{x}),$$

where,

$$h_{ij}^{(k)}(t, \mathbf{x}) = \frac{1}{R^2(t)} Y_k(t) \zeta_{ij}(\mathbf{k}, \mathbf{x}).$$

The functions $Y_k(t)$ and $\zeta_{ij}(\mathbf{k}, \mathbf{x})$ obey the following equations:

$$\left(\nabla^2 + k^2\right) \zeta_{ij} = 0,$$

where $\nabla^2$ is the usual Laplacian, and

$$\ddot{Y}_k(t) + f(t)\dot{Y}_k(t) + g(t)Y_k(t) = 0, \tag{17}$$

where,

$$f(t) = \frac{\dot{\phi}}{\phi} - H, \tag{18}$$

and,

$$g(t) = \frac{k^2}{R^2} - 4H^2 - \frac{8\pi}{\phi}\left[\frac{2(1+\omega)}{2\omega+3}\rho - \frac{2\omega}{2\omega+3}p\right] - \frac{d\omega/d\phi}{2\omega+3}\frac{\dot{\phi}^2}{\phi} - 2\frac{\phi(d\Lambda/d\phi)+2(\omega+1)\Lambda}{2\omega+3}. \tag{19}$$

As usual, $H$ stands for the Hubble's parameter, and

$$k = |\mathbf{k}| = 2\pi\frac{R}{\lambda}$$

($\mathbf{k}$ is the comoving wave vector, and $\lambda$ is the wavelength).

The amplitude $Y_k(t)$ has to increase, for increasing age of the Universe ($t$), in order to obtain sustained gravitational waves. Its equation can be cast into the following one:

$$\ddot{Y}_k + \left[\frac{\dot{\phi}}{\phi} - H\right]\dot{Y}_k + \left[\frac{k^2}{R^2} - 2\frac{\ddot{R}}{R} - 2\frac{\dot{\phi}}{\phi}H - \frac{1}{(2\omega+3)}\frac{\dot{\phi}^2}{\phi}\frac{d\omega}{d\phi} - 2\frac{\phi\,(d\Lambda/d\phi)+2\,(\omega+1)\,\Lambda}{2\omega+3}\right]Y_k = 0 \tag{20}$$

When we plug the data obtained in the last section, we shall find a solution, involving the conditions,

$$1 - \alpha > 0, \tag{21}$$

and,

$$2 + 3\alpha < 0, \tag{22}$$

such that, the term $g(t)$ is dominant, exponentially increasing, but negative,

$$\begin{aligned} g(t) &\cong 4H^2 - \frac{8\pi\rho_0}{\phi_0}\,(1-\alpha)\,e^{-H(2+3\alpha)t} - \frac{1}{2}H^2\,(2+3\alpha) \\ &\quad - 2\Lambda_0 e^{-3H(1+\alpha)t} - 3\,(1+\alpha)\,\frac{\Lambda_0}{\omega_0}e^{-Ht} \\ &\approx -\frac{8\pi\rho_0}{\phi_0}\,(1-\alpha)\,e^{-H(2+3\alpha)t}, \end{aligned} \tag{23}$$

and the equation for the amplification is approximately given by,

$$\ddot{Y}_k \cong 2H\dot{Y}_k + \frac{8\pi\rho_0}{\phi_0}\,(1-\alpha)\,e^{-H(2+3\alpha)t}\,Y_k, \tag{24}$$

which denotes an exponentially increasing amplification.

For instance, consider that the last exponential term in (24) is equal to 1 (i.e., $2+3\alpha \cong 0$). Then, the solution of the amplification is given by,

$$Y_k \cong Be^{\beta t} \quad (B, \beta = \text{positive constants}).$$

A quick analysis shows that the amplified signal, which is given by the perturbation,

$$h_k = \frac{Y_k}{a^2},$$

also increases exponentially, because,

$$\beta \cong H + \sqrt{H^2 + \frac{8\pi\rho_0}{\phi_0}(1-\alpha)} > 2H. \tag{25}$$

In any case, the dominance of $g(t)$, guarantees that $Y_k$ grows exponentially. Amplification of gravitational waves has been thus, proved.

## 4 Conclusion

The results obtained here, limit the possible equations of state; in General Relativity, we take exponential inflation with $\alpha = -1$. Here, the only limitations are given by conditions (21) and (22). As these limitations are not out-of-question, we consider our solutions very reasonable, and we advance the conclusion that gravitational waves are indeed amplified by inflation. In Berman [8], the static Universe ($\dot{a} = 0$) is also considered: the conclusion of that paper was that amplification of gravitational waves is not restricted to evolutionary models.

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